

# Jammed disks in narrow channel: criticality and ordering tendencies

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A system of identical disks is confined to a narrow channel, closed off at one end by a stopper and at the other end by a piston. All surfaces are hard and frictionless. A uniform gravitational field is directed parallel to the plane of the disks and perpendicular to the axis of the channel. We employ a method of configurational statistics that interprets jammed states as configurations of floating particles with structure. The particles interlink according to set rules. The two jammed microstates with smallest volume act as pseudo-vacuum. The placement of particles is subject to a generalized Pauli principle. Jammed macrostates are generated by random agitations and specified by two control variables. One is a measure of the intensity of random agitations at given pressure. The other is a measure of the change in gravitational potential energy in units of compression work when one particle is excited. In this two-dimensional space of variables there exists a critical point. The jammed macrostate realized at the critical point depends on the path of approach. We describe all jammed macrostates by volume and entropy. Both are functions of the average population densities of particles. Approaching the critical point in an extended space of control variables generates two types of jammed macrostates: states with random heterogeneities in mass density and states with domains of uniform mass density.

## I. INTRODUCTION

Granular matter consists of particles with sizes too big for thermal fluctuations to have a significant impact. The spatial configurations of grains depends on many factors including their sizes and shapes, interactions between them, interactions with external fields, and a protocol for producing them in a controllable way [1–3]. Models of granular matter tend to omit non-essential attributes of grains in efforts to gain more profound knowledge of bulk behavior.

In this spirit, jammed states of rigid objects with no interactions other than hard-core repulsion have become the focus of numerous studies. It has become common to describe jammed states of granular matter in the language of equilibrium statistical mechanics albeit with provisos. We shall use the term *configurational statistics* for this framework of analysis [4–6]. Jammed states are frozen, hence time averages useless. Nevertheless, jammed macrostates are postulated to exist, to be systematically reproducible by means of some protocol of random agitations, and to be describable by entropies and averaged densities in spatial regions of macroscopic size.

To the many interesting themes surrounding jammed granular matter belong the existence and nature of critical singularities and the onset of ordering [7]. In this work we use a very simple scenario that enables us to investigate this theme with some degree of rigor.

Inspired by prior work [8–10], we consider a long, narrow channel of width  $H$  with the axis in a horizontal direction, a stopper at one end, and a piston at the other end as shown in Fig. 1. The channel contains  $N$  disks of mass  $m$  and diameter  $\sigma$ . All surfaces are rigid and frictionless. Jamming requires three points of contact on each disk that are not on the same semicircle. A uniform gravitational field  $g$  in the direction shown is present.

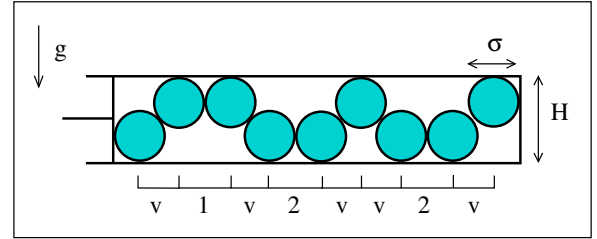


FIG. 1: Jammed microstate of disks of mass  $m$  and diameter  $\sigma$  in a channel of width  $H$  with rigid walls, stopper, and piston. For  $1 < H/\sigma < 1 + \sqrt{3}/4$  all configurations are sequences of four distinct interlinking two-disk tiles. In our methodology, two tiles (marked  $v$ ) are elements of pseudo-vacuum and the other two (marked 1,2) are particles.

In channels with  $1 < H/\sigma < 1 + \sqrt{3}/4$ , every jammed disk (except the two outermost) touches one wall and two adjacent disks. Every possible jammed microstate is a sequence of interlinking two-disk tiles. The four distinct tiles are identified in Fig. 1. The two most compact states consists of tiles  $v$  in alternating sequence [11]. The least compact states are sequences  $[v1v2v1v2 \dots v]$  and  $[v2v1v2v1 \dots v]$ . Only tiles  $v$  can follow each other directly. Any tile 1 (2) is separated from the next tile 1 (2) by at least two tiles  $v$ . A tile 1 is separated from the nearest tile 2 by at least one tile  $v$ .

In channels with  $1 + \sqrt{3}/4 < H/\sigma < 2$ , there exist 32 tiles interlinking according to more complicated rules [9]. If  $H/\sigma > 2$  disks can pass one another with the consequence that the jamming condition becomes nonlocal. Here we assume that  $H/\sigma < 1 + \sqrt{3}/4$ .

Jammed microstates in this system are countable. Each has a well defined volume. Many have the same volume. They represent one jammed macrostate in the sense of an ensemble average. The multiplicity of mi-

crostates with given volume determines the configurational entropy of that macrostate. In the limit  $N \rightarrow \infty$  the entropy per disk,  $\bar{S} \doteq S/N$  becomes a smooth function of reduced excess volume,  $\bar{V} \doteq (V - V_0)/N$ , where  $V_0$  is the volume of the most compact jammed state.

The inverse slope of the configurational entropy curve is an intensive thermodynamic variable named *compactivity* [4]:

$$X \doteq \left( \frac{d\bar{S}}{d\bar{V}} \right)^{-1}. \quad (1)$$

This definition works well when a jammed macrostate is specified by a single intensive variable. Typical for such cases is that  $\bar{S}$  increases monotonically between  $\bar{V} = 0$  and some value  $\bar{V} = \bar{V}_\infty$ . The function  $\bar{S}(\bar{V})$  is concave along that stretch with infinite initial slope and zero final slope. The compactivity thus varies between  $X = 0$  at  $\bar{V} = 0$  and  $X = \infty$  at  $\bar{V} = \bar{V}_\infty$ . It is a measure for the intensity by which the system is randomly agitated at given pressure of the piston to produce a specific jammed macrostate. A dosage of intense random agitations produces a highly compactifiable state with much excess volume. Jolting the system with less intensity produces a more compact and less compactifiable jammed macrostate.

Between the most disordered jammed macrostate at  $\bar{V} = \bar{V}_\infty$  and the least compact jammed macrostate at  $\bar{V} = \bar{V}_{max}$  the function  $\bar{S}(\bar{V})$  stays concave and thus decreases. These jammed macrostates have negative compactivity. They do exist but cannot, in general, be generated by random agitations.

This scenario is borne out if we set  $g = 0$  in our system or tilt the channel into a horizontal plane. The configurational entropy of this case was determined in [8]. The gravitational field  $g$  complicates matters significantly. Jammed macrostates are now specified by two independent intensive variables. Microstates with equal volume do no longer have equal statistical weight on account of their different gravitational potential energy.

The methodology is further developed (Sec. II).  $\bar{S}$  turns out to be a multiple-valued function of  $\bar{V}$  (Sec. III). In the space of the two independent intensive variables there exists a critical point (Sec. IV). Ordering tendencies at criticality are explored in a space of three independent intensive variables (Sec. V). More complex scenarios now appear within reach of rigorous analysis (Sec. VI).

## II. METHODOLOGY

Here we adapt a method of statistical mechanical analysis based on fractional exclusion statistics to jammed states of granular matter. This approach was originally developed for quantum many-body systems [12–15] and was recently extended to classical particles with shapes [16–19]. Whereas microscopic degrees of freedom are subject to thermal fluctuations, the disks in the narrow channel need to be subjected to random agitations to produce

a comparable effect. In the context of this study the disks are not the particles themselves. The particles (or quasiparticles) considered here are the tiles 1 and 2. The method to be described has a combinatorial part and a statistical mechanical part. We begin with a discussion of the variables that we employ to describe jammed macrostates.

### A. Energetics

The volume of each element of pseudo-vacuum as indicated by a bracket in Fig. 1 is  $\sigma\sqrt{1 - (H/\sigma - 1)^2}$  and the volume of a particle from either species is  $\sigma$ . Hence the activation of each particle increases the volume by a fraction of the disk diameter,

$$v_1 = v_2 = q\sigma, \quad q \doteq 1 - \sqrt{1 - (H/\sigma - 1)^2}. \quad (2)$$

The activation of a particle 1 or 2 adds energy. One disk moves up or down and the piston moves out:

$$\epsilon_{1,2} = pq\sigma \pm \gamma, \quad \gamma = mg(H - \sigma). \quad (3)$$

We shall see that the configurational entropy,  $\bar{S}(\bar{V})$ , now depends on a parameter that contains the ratio  $p/g$ . Jammed macrostates are specified by two independent intensive variables. We get to them naturally, in the framework of configurational statistics, if we start from the general thermodynamic relation  $(\partial\bar{S}/\partial\bar{V})_{\bar{U}} = p/T$ .

For  $g = 0$  this is the inverse of compactivity (1), the only intensive variable needed. In a thermal system, the dimensionless quantity,

$$K_p \doteq \frac{pq\sigma}{k_B T}, \quad (4)$$

compares a unit of thermal energy with a unit of compression work. For the *athermal* system of jammed disks this translates into a measure of the intensity of random agitations performed at a given pressure of the piston. The jammed macrostate resulting from random agitations only depends on that ratio.

For  $g \neq 0$  a second dimensionless quantity comes into play. In a thermal system the quantity,

$$K_g \doteq \frac{\gamma}{k_B T}, \quad (5)$$

compares the same unit of thermal energy with a convenient unit of gravitational potential energy. For the jammed disks this translates into a different measure of intensity of random agitations, performed in a given gravitational field. Configurational statistics postulates that jammed macrostates are reproducibly characterized by  $K_p$  and  $K_g$ .

The thermodynamic variables  $T$  and  $p$  per se have no meaning for the description of jammed macrostates. Nevertheless, in an experiment, random agitations are performed at some piston pressure. The resulting jammed

macrostate depends on the intensity of random agitations and on how that pressure compares with the weight of the disks. The entropy  $\bar{S}$  and the excess volume  $\bar{V}$  are functions of  $K_p$  and  $K_g$ , but the shape of the function  $\bar{S}(\bar{V})$  inferred from them only depends on the experimentally controllable parameter,

$$\Gamma \doteq \frac{K_g}{K_p} = \frac{\gamma}{pq\sigma}. \quad (6)$$

## B. Combinatorics

For the combinatorial analysis we use the template designed for statistically interacting particles familiar from previous work (adapted to open boundary conditions) [12, 17]:

$$W(\{N_m\}) = n_{pv} \prod_{m=1}^M \binom{d_m + N_m - 1}{N_m}, \quad (7a)$$

$$d_m = A_m - \sum_{m'=1}^M g_{mm'}(N_{m'} - \delta_{mm'}). \quad (7b)$$

This multiplicity expression yields the number of microstates with given particle content. The degeneracy of the pseudo-vacuum is encoded in  $n_{pv}$ . The  $A_m$  are capacity constants and the  $g_{mm'}$  are statistical interaction coefficients.

All particles of a given species considered here will have the same volume and energy no matter where they are placed. Therefore, all microstates with given particle content constitute a macrostate in the sense discussed earlier. The entropy of a macrostate as derived from the multiplicity expression (7) for  $N_m \gg 1$  via  $S = k_B \ln W$  reads [14, 18]:

$$S(\{N_m\}) = k_B \sum_{m=1}^M \left[ (N_m + Y_m) \ln (N_m + Y_m) - N_m \ln N_m - Y_m \ln Y_m \right], \quad (8a)$$

$$Y_m \doteq A_m - \sum_{m'=1}^M g_{mm'} N_{m'}. \quad (8b)$$

## C. Statistical mechanics

The statistical mechanical analysis for a thermal system [13, 14] produces, for the grand partition function, the result [17]

$$Z = \prod_{m=1}^M (1 + w_m^{-1})^{A_m}, \quad (9)$$

where the (real, positive)  $w_m$  are the solutions of the coupled nonlinear algebraic equations,

$$e^{K_m} = (1 + w_m) \prod_{m'=1}^M (1 + w_{m'}^{-1})^{g_{m'm}}, \quad (10)$$

and where  $K_m \doteq \epsilon_m/k_B T$ . The average population densities,  $\bar{N}_m \doteq \langle N_m \rangle / N$ , of particles from each species are derived from the coupled linear equations,

$$w_m \bar{N}_m + \sum_{m'=1}^M g_{mm'} \bar{N}_{m'} = \bar{A}_m, \quad (11)$$

where  $\bar{A}_m \doteq A_m/N$ . For jammed-disk macrostates the  $K_m$  are linear combinations of (4) and (5). The configurational entropy,  $\bar{S}(\bar{V})$ , follows directly from a scaled version of the entropy expression (8) in combination with the volume expression [20],

$$\bar{V}(\{\bar{N}_m\}) = \sum_{m=1}^M \bar{N}_m v_m. \quad (12)$$

## III. CONFIGURATIONAL ENTROPY

All jammed microstates such as the one shown in Fig. 1 contain particles from  $M = 2$  species. The specifications needed for the statistical mechanical analysis are listed in Table I. In the taxonomy of [17] both species are compacts.

TABLE I: Specifications of the particles represented by tiles 1 and 2: species, volume and energy (relative to vacuum), capacity constant (left), and interaction coefficients (right). The elements of vacuum have (absolute) volume  $(1 - q)\sigma$  and energy  $p(1 - q)\sigma$ .

$m$	$v_m$	$\epsilon_m$	$A_m$	$g_{mm'}$	1	2
1	$q\sigma$	$q\sigma p + \gamma$	$\frac{1}{2}(N - 3)$	1	$\frac{3}{2}$	$\frac{1}{2}$
2	$q\sigma$	$q\sigma p - \gamma$	$\frac{1}{2}(N - 3)$	2	$\frac{1}{2}$	$\frac{3}{2}$

### A. Zero gravity

In the special case  $g = 0$ , particles 1 and 2 have equal energy. No distinction is necessary. They can be merged into a single species  $\bar{1}$  with specifications  $A_{\bar{1}} = N - 3$  and  $g_{\bar{1}\bar{1}} = 2$ . It is a type-1 merger in the classification of [19]. The configurational entropy follows directly from (8) with these specifications:

$$\bar{S} = k_B \left[ (1 - \bar{N}_{\bar{1}}) \ln(1 - \bar{N}_{\bar{1}}) - \bar{N}_{\bar{1}} \ln \bar{N}_{\bar{1}} - (1 - 2\bar{N}_{\bar{1}}) \ln(1 - 2\bar{N}_{\bar{1}}) \right], \quad (13)$$

where  $\bar{V} = q\sigma\bar{N}_1$ . This result was first derived in [8] by a different method. In this case we do not have to solve (10) and (11) to get to  $\bar{S}(\bar{V})$ . Those solutions are

$$w_1 = \frac{1}{2}e^{K_1} \left[ 1 + \sqrt{1 + 4e^{-K_1}} \right], \quad \bar{N}_1 = \frac{1}{w_1 + 2}, \quad (14)$$

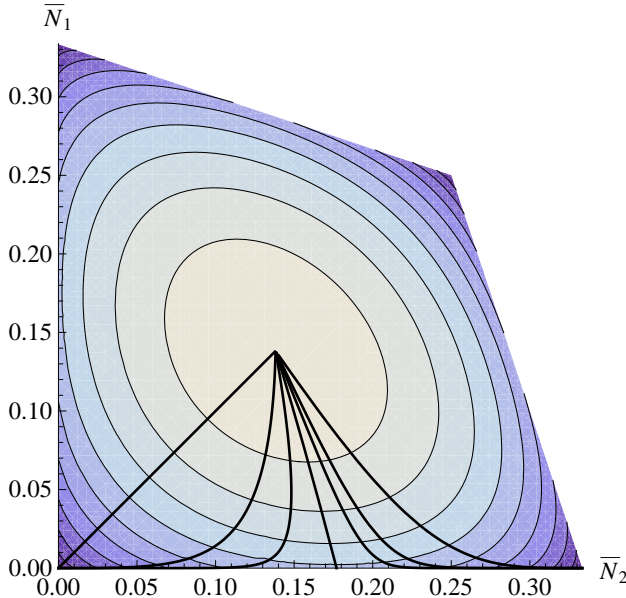
where  $K_1 = K_p$  from (4). The entropy inferred from (9) reads

$$\bar{S} = k_B \left[ \ln(1 + w_1^{-1}) + \frac{K_1}{2 + w_1} \right]. \quad (15)$$

Expressions (14) and (15) are consistent with (13). All the attributes of  $\bar{S}(\bar{V})$  described in Sec. I are here realized [21].

### B. Nonzero gravity

For  $g \neq 0$  the entropy depends on two independent variables. The function  $\bar{S}(\bar{N}_1, \bar{N}_2)$  inferred from (8) with the  $A_m$  and  $g_{mm'}$  from Table I produces an entropy landscape as shown in Fig. 2. It is of quadrilateral shape with a smooth maximum in the center and zeros at the corners. All jammed states on a line  $\bar{N}_1 + \bar{N}_2 = \text{const}$  have the same volume but different gravitational potential energies. Expression (13) is recovered by  $\bar{S}(\bar{N}_1/2, \bar{N}_1/2)$ . The system has the highest capacity for particles if both species are present in equal numbers.



macrostate with the highest possible entropy. All curves for nonzero gravity start from that highest-entropy point and fan out in different directions.

At  $\Gamma < 1$  both species of particles have positive activation energies. The physics does not change qualitatively from the zero-gravity case. The gravitational field suppresses the population density of particles 1 and enhances that of particles 2. The net effect is a modest reduction of entropy (of mixing) at given volume. All curves end in the most compact jammed state.

At  $\Gamma > 1$ , the particles from species 2 have negative activation energies. The jammed state produced by random agitations in the low-intensity limit is now qualitatively different. It is a periodic array of close-packed particles 2. It has volume  $\bar{V}/q\sigma = \frac{1}{3}$ , which is larger than the ( $\Gamma$ -independent) jammed state produced in the high-intensity limit of random agitations.

Increasing the intensity of random agitation always increases the entropy. However, at  $\Gamma > 1$ , it initially decreases the volume of the states thus produced. The decreasing  $\bar{V}$  is associated with compression work used for weight lifting. The annihilation of particles 2 creates disorder and thus increases  $\bar{S}$ . At higher intensities the jammed states have larger volume again. Here particles 1, which have high energy, are created in significant numbers.

For the borderline case  $\Gamma = 1$  between the two regimes, the particles from species 2 have zero activation energy. Gravitational energy and expansion energy are in balance. Random agitations do not produce ordering in the low-intensity limit. The shapes of the curves in Fig. 3 signal the presence of a critical singularity.

#### IV. CRITICALITY

In the framework of configurational statistics it is legitimate to use the term *criticality* more loosely than is common in the theory of (thermal) phase transitions. In our system the critical singularity is associated with the combined limit

$$K_p \rightarrow \infty, \quad K_g \rightarrow \infty, \quad \Gamma \rightarrow 1. \quad (18)$$

The jammed macrostate realized at the critical singularity depends on how the singularity is approached. In Sec. III we have considered one particular approach, with  $\Gamma = 1$ , implying that  $w_1 \rightsquigarrow \infty$  and

$$w_2 \rightsquigarrow \frac{(9 - \sqrt{69})^{1/3} + (9 + \sqrt{69})^{1/3}}{2^{1/3} 3^{2/3}} = 1.3247\dots, \quad (19)$$

the physically relevant solution of the cubic equation,

$$w_2^3 - w_2 - 1 = 0. \quad (20)$$

The associated values for excess volume and entropy are

$$\bar{S}/k_B \rightsquigarrow \ln w_2 = 0.28119\dots, \quad (21a)$$

$$\bar{V}/q\sigma \rightsquigarrow (3 + 2w_2)^{-1} = 0.17700\dots \quad (21b)$$

Further critical macrostates are represented by points on the (dashed) arching line in Fig. 3 and by points in the space underneath. The latter will be discussed in Sec. V. The former are realized in a family of pathways toward the critical singularity specified by a single parameter,

$$\Delta \doteq K_p - K_g. \quad (22)$$

The critical macrostates are determined from

$$w_1 = \infty, \quad e^{2\Delta} = \frac{w_2^3}{1 + w_2}, \quad -\infty < \Delta < +\infty \quad (23)$$

and the particle population densities (11) become

$$\bar{N}_1 = 0, \quad \bar{N}_2 = \frac{1}{3 + 2w_2}. \quad (24)$$

The point  $\Delta = 0$  recovers the case  $\Gamma = 1$  discussed previously.

Only particles 2 are present. They are distributed randomly along the channel. The volume and the entropy depend on the population density  $\bar{N}_2$ :

$$\bar{V}/q\sigma = \bar{N}_2 \quad (25a)$$

$$\begin{aligned} \bar{S}/k_B = & \frac{1 - \bar{N}_2}{2} \ln \left( \frac{1 - \bar{N}_2}{2} \right) - \bar{N}_2 \ln \bar{N}_2 \\ & - \frac{1 - 3\bar{N}_2}{2} \ln \left( \frac{1 - 3\bar{N}_2}{2} \right). \end{aligned} \quad (25b)$$

Complete ordering exists if there are no particles or if the particles are close packed. The function  $\bar{S}(\bar{V})$  has zero slope at the point (21) and infinite slope at  $\bar{S} = 0$ :

$$\frac{d\bar{S}}{d\bar{V}} \rightsquigarrow \begin{cases} -\ln \bar{V}, & \bar{V} \rightarrow 0 \\ -\frac{3}{2} \ln \left( \frac{1}{3} - \bar{V} \right), & \bar{V} \rightarrow \frac{1}{3} \end{cases}. \quad (26)$$

All critical macrostates are averages of the same subset of jammed microstates, but weighted differently. The macrostate at the top of the arch has all microstates from that subset weighted equally. The macrostates elsewhere on the arch result from averages with one bias in the weighting. The parameter  $\Delta$  controls the population density  $\bar{N}_2$  and thus the volume. All microstates with a given number of particles 2 remain weighted equally. The macrostates inside the arch result from averages with two biases in the weighting.

#### V. ORDERING TENDENCIES

We choose a second bias in the form of a weak interaction force that either enhances or suppresses heterogeneities in mass density along the channel. It controls the degree of clustering of particles 2. New pathways to the critical point generate macrostates with domains of uniform mass density. No symmetry-breaking field needs to be introduced to make that happen.

We extend the set of particle species from two to three. We split one species into two: every compact 2 in the system becomes either a host 2' or a tag 2''. The particle 2 closest to the piston is a host 2'. Any other particle 2 is a tag 2'' if it follows the preceding particle 2 in one of the two shortest tile sequences: 2vv2 or 2v1v2. Otherwise it is a host again.

In Fig. 1 we have one host 2' followed by a tag 2'' (left to right). We do not split compacts 1 because they become depleted as criticality is approached. The specifications of the three species, compact 1, host 2', and tag 2'', are compiled in Table II. Anghel's rules [15] are satisfied.

TABLE II: Specifications of three species of particles: species, volume and energy (relative to vacuum), capacity constant (left), and statistical interaction coefficients (right).

$m$	$v_m$	$\epsilon_m$	$A_m$	$g_{mm'}$	1	2'	2''
1	$q\sigma$	$pq\sigma + \gamma$	$\frac{1}{2}(N-3)$	1	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
2'	$q\sigma$	$pq\sigma - \gamma$	$\frac{1}{2}(N-3)$	2'	$\frac{1}{2}$	$\frac{5}{2}$	$\frac{3}{2}$
2''	$q\sigma$	$pq\sigma - \gamma - \phi$	0	2''	0	-1	0

Raising (lowering) the activation energy of the tag relative to that of the host amounts to a repulsive (attractive) short-range force between particles 2. Such a force suppresses (enhances) heterogeneities in mass density. It opens up new pathways to the critical singularity. In the critical macrostates thus generated the microstates with a given number of particles 2 are no longer weighted equally. The weighting now discriminates between hosts and tags.

The evidence for the formation of domains presented here is indirect, as found in the entropy [22]. Assembling the function  $\bar{S}(\bar{V})$  in the extended parameter space starts from the extended Eqs. (16),

$$e^{2K_p} = \frac{w_1^2 w_{2'}^3 (1 + w_{2''})}{(1 + w_1)(1 + w_{2'})^2 w_{2''}},$$

$$e^{2K_g} = \frac{w_1(1 + w_{2'})w_{2''}}{w_{2'}^2(1 + w_{2''})}, \quad e^\Phi = \frac{w_{2'}}{w_{2''}}, \quad (27)$$

where  $\Phi \doteq \phi/k_B T$  is an amendment to (4) and (5). The population densities of compacts 1, hosts 2', and tags 2'' are

$$\bar{N}_1 = \frac{1 + 2w_{2''} + w_{2'}w_{2''}}{D'},$$

$$\bar{N}_{2'} = \frac{(1 + w_1)w_{2''}}{D'}, \quad \bar{N}_{2''} = \frac{1 + w_1}{D'}, \quad (28)$$

$$D' = 4 + 3w_1 + 7w_{2''} + 5w_1w_{2''} + 3w_{2'}w_{2''} + 2w_1w_{2'}w_{2''}.$$

We now have a two-parameter family of pathways toward the critical singularity with critical macrostates de-

termined from

$$w_1 = 0, \quad e^{2\Delta} = \frac{w_{2'}^5(1 + w_{2''})^2}{(1 + w_{2'})^3 w_{2''}}, \quad e^\Phi = \frac{w_{2'}}{w_{2''}}, \quad (29)$$

$$-\infty < \Delta < +\infty, \quad -\infty < \Phi < +\infty,$$

and with critical population densities,

$$\bar{N}_1 = 0, \quad \bar{N}_{2'} = w_{2''} \bar{N}_{2''} = \frac{w_{2''}}{3 + 5w_{2''} + 2w_{2'}w_{2''}}. \quad (30)$$

The function  $\bar{S}(\bar{N}_{2'}, \bar{N}_{2''})$  inferred from (8) is shown in Fig. 4. Every point in this entropy landscape represents a macrostate at criticality. All critical macrostates have  $pq\sigma = \gamma$  and  $\phi = 0$ .

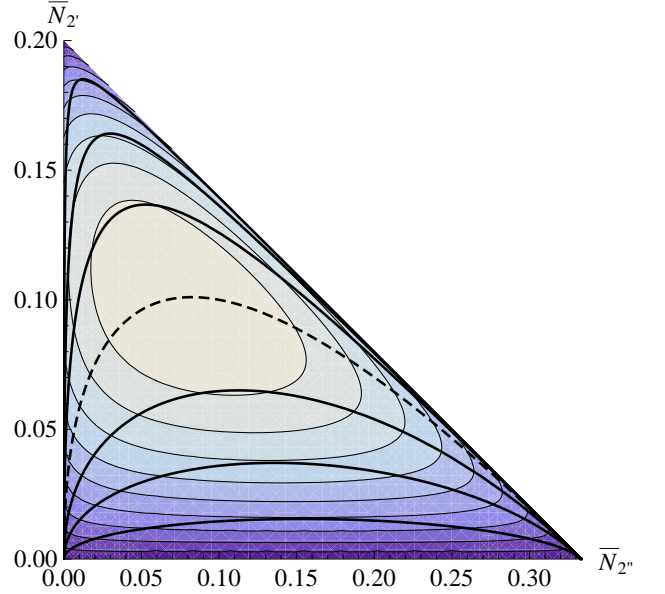


FIG. 4: Entropy per disk,  $\bar{S}/k_B$ , versus population densities  $\bar{N}_{2''}, \bar{N}_{2'}$  of particles from the two species 2'' and 2'. The contours are at  $0.028\ell, \ell = 1, \dots, 9$ . The solid and dashed curves represent (30) for  $\Phi = 0, \pm 1.5, \pm 3.0, \pm 5.0$  in ascending order from top to bottom.

In macrostates pertaining to  $\Phi = 0$ , particles 2 form random clusters. They are the states of highest entropy at given  $\bar{N}_{2'} + \bar{N}_{2''}$ , represented by the dashed curve in Fig. 4. This particular mix of hosts and tags,

$$\bar{N}_{2'} = \frac{1}{4} \left[ \sqrt{\bar{N}_{2''}(8 + \bar{N}_{2''})} - 5\bar{N}_{2''} \right], \quad (31)$$

produces a characteristic texture of heterogeneity in mass density. Critical macrostates on either side of the dashed curve have mass heterogeneities with textures that reflect a higher degree of ordering.

In macrostates generated on pathways with  $\Phi > 0$  tags are more abundant and hosts less abundant than in a mixture of randomly placed particles 2. The bottom three solid curves are critical population densities (30) at

fixed  $\Phi > 0$ . Clusters of hosts with tags are domains of uniform, low mass density, phase separated from domains of uniform, high mass density. The latter are clusters of tiles  $v$ .

These domains suppress the entropy significantly relative to the entropy of random clusters of particles 2. The entropy reduction is demonstrated by the contours in Fig 4 and, more compellingly, in the configurational entropy curves,  $\bar{S}(\bar{V})$ , shown in Fig. 5(a). The effect is most significant at intermediate volume where large clusters of tiles 2 or tiles  $v$  are least likely to be realized by chance.

A different ordering tendency is manifest in critical macrostates generated along pathways with  $\Phi < 0$ . More hosts and fewer tags are present than in a randomly placed mixture. Critical population densities (30) at fixed  $\Phi < 0$  are represented by the top three solid curves in Fig. 4. These macrostates exhibit various degrees of dispersal of particles 2. The dispersal flattens out heterogeneities in mass density and thus reduces the entropy.

This ordering tendency manifests itself differently in the configurational entropy as shown in Fig. 5(b). Dis-

persal of particles 2 only produces ordering tendencies in conjunction with spatial constraints. The entropy reduction remains insignificant until crowding of particles 2 becomes an issue. The capacity of the channel for hosts 2' is only 60% of its capacity for particles 2. Hence, hosts alone reach saturation at  $\bar{V}/q\sigma = \frac{1}{5}$  in an ordered state. At larger  $\bar{V}$ , the entropy rises again on account of tags attached to some hosts.

## VI. DISCUSSION AND OUTLOOK

The methodology of this work builds on the foundations of configurational statistics. Chief among the assumptions is the existence of jammed macrostates that are systematically reproducible by some protocol of random agitations of given intensity and that can be described by few control variables.

Jammed macrostates of the system under investigation here are parametrized by two intensive variables reflecting the intensity of the random agitations in two distinct energy units of relevance. One unit represents the compression work of the piston when it is moved a fraction of one disk diameter. The other unit represents the weight of one disk lifted across the channel.

A protocol of random agitations that fits the bill of configurational statistics for the jammed disks in the narrow channel is not guaranteed to exist. If it does exist it may be difficult to implement. It is no secret that this is the Achilles' heel of configurational statistics.

The detailed predictions of this work, which follow rigorously from the assumptions underlying configurational statistics, present a situation where these very assumptions can be put to the test of simulations and experiments. The simplicity of the model in conjunction with the complexity of its behavior is well suited for that purpose. Any instances of verification or falsification will shed light on the important question of how accurately a protocol of random agitations applied to jammed granular matter can mimic the effects of thermal fluctuations.

The methodology which combines combinatorial analysis and statistical mechanical analysis in an adaptation to jammed granular matter as presented here is amenable to further development in several directions: (i) the regime  $1 + \sqrt{3/4} < H/\sigma < 2$  of 32 tiles as identified in [9], (ii) mixtures of disks with different radii and masses including rattlers [23], (iii) spatial correlations of heterogeneities in mass density, and (iv) channels with the axis oriented vertically.

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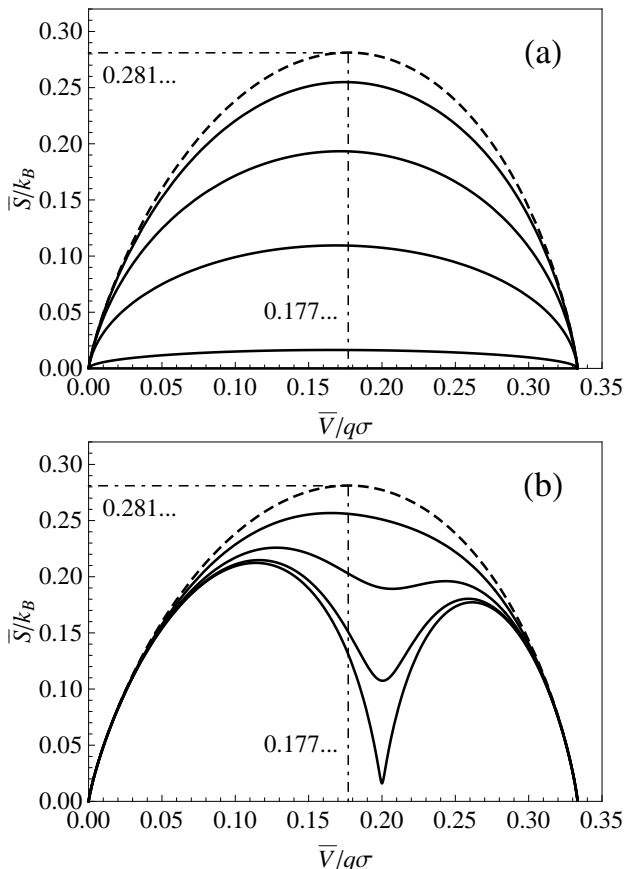


FIG. 5: Entropy per disk versus excess volume at criticality. The five curves, inferred from (29), represent, from top to bottom, critical macrostates characterized (a) by  $\Phi = 0, 1.5, 3, 5, 10$ . and (b) by  $\Phi = 0, -1.5, -3, -5, -10$

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  - [21] Contact with quantities in Table I of [4] are readily established. If we set  $\tilde{X} \doteq K_1^{-1}$ ,  $\tilde{V} \doteq \tilde{V}/q\sigma$ ,  $\tilde{S} \doteq \tilde{S}/k_B$ , and  $w \doteq w_1$ , implying  $\tilde{V} = (2+w)^{-1}$ ,  $e^{1/\tilde{X}} = w^2/(1+w)$ ,  $\tilde{S} = \ln(1+w^{-1}) + \tilde{V}/\tilde{X}$ , and  $\tilde{Z} = (1+w^{-1})$ , we obtain  $\tilde{Y} \doteq \tilde{V} - \tilde{X}\tilde{S} = -\tilde{X} \ln(1+w^{-1})$  and, hence,  $\tilde{Z} = e^{-\tilde{Y}/\tilde{X}}$ .
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